Exam Calculus 2

6 April 2021, 18:45-21:45

The exam consists of 6 problems. You can achieve 100 points which includes a bonus of 10 points. You have 180 minutes to answer the questions. Students eligible for extra time get 30 minutes extra. In addition, all students have 15 minutes to scan and upload the solutions to Nestor. Only handwritten solutions will be accepted. Upload your solutions in a single file. For the filename, use the format Lastname_Studentnumber_Exam. **Each problem has a version A and a version B. Which version you need to do depends on your student number** $\mathsf{s\,n_1 n_2 n_3 n_4 n_5 n_6 n_7}$ **as explained in the problem.**

1. [**5+5+5=15 Points**] Do version **A** if $\mathsf{n_7}$ is odd and **B** if $\mathsf{n_7}$ is even, where $\mathsf{s\,n_1 n_2 n_3 n_4 n_5 n_6 n_7}$ is your student number.

   Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$\textbf{A: } f(x,y) = \begin{cases} \frac{x^2|y|}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases},$$

$$\textbf{B: } f(x,y) = \begin{cases} \frac{|x|y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

   (a) Is f continuous at $(x,y) = (0,0)$? Justify your answer.

   (b) Use the definition of directional derivatives to determine for which unit vectors $\boldsymbol{u} = (v,w) \in \mathbb{R}^2$ the directional derivative $D_{\boldsymbol{u}} f(0,0)$ exists.

   (c) Is f differentiable at $(x,y) = (0,0)$? Justify your answer.

2. [**7+8=15 Points**] Do version **A** if $\mathsf{n_6}$ is odd and **B** if $\mathsf{n_6}$ is even, where $\mathsf{s\,n_1 n_2 n_3 n_4 n_5 n_6 n_7}$ is your student number, where $\mathsf{s\,n_1 n_2 n_3 n_4 n_5 n_6 n_7}$ is your student number.

   Suppose the function $f : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$ is of class C^2. Substituting $x(s,t)$ and $y(s,t)$ in $f(x,y)$ defines a function $w(s,t)$.

   (a) Show that

$$\textbf{A:} \qquad w_{ss} = f_{xx}\,(x_s)^2 + 2 f_{xy}\, x_s\, y_s + f_{yy}\,(y_s)^2 + f_x\, x_{ss} + f_y\, y_{ss}\,,$$
$$\textbf{B:} \qquad w_{tt} = f_{xx}\,(x_t)^2 + 2 f_{xy}\, x_t\, y_t + f_{yy}\,(y_t)^2 + f_x\, x_{tt} + f_y\, y_{tt}\,,$$

   where the function arguments are omitted to avoid a lengthy notation.

   (b) For

$$\textbf{A: } (x(s,t), y(s,t)) = (\mathrm{e}^s \cos t, \mathrm{e}^s \sin t) \qquad \textbf{B: } (x(s,t), y(s,t)) = \Big(\frac{s}{s^2+t^2}, \frac{-t}{s^2+t^2}\Big),$$

   show that if $f_{xx} + f_{yy} = 0$ then $w_{ss} + w_{tt} = 0$.

3. **[3+6+6=15 Points]** Do version **A** if n_5 is odd and **B** if n_5 is even, where $\mathsf{s}\,\mathsf{n}_1\mathsf{n}_2\mathsf{n}_3\mathsf{n}_4\mathsf{n}_5\mathsf{n}_6\mathsf{n}_7$ is your student number.

   Let S be the ellipsoid

$$\textbf{A: } S = \{(x,y,z) \mid \frac{x^2}{9} + \frac{y^2}{4} + z^2 = 3\} \qquad \textbf{B: } S = \{(x,y,z) \mid \frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3\}$$

   and $(x_0, y_0, z_0) \in S$ be the point

$$\textbf{A: } (x_0, y_0, z_0) = (3, -2, 1) \qquad \textbf{B: } (x_0, y_0, z_0) = (2, 1, -3)\,.$$

   (a) Determine the tangent plane of S at (x_0, y_0, z_0).

   (b) Use the Implicit Function Theorem to show that near the point (x_0, y_0, z_0), the ellipsoid S can be considered to be the graph of a function f of y and z. Compute the partial derivatives of f with respect to y and z and show that the tangent plane found in (a) coincides with the graph of the linearization of f at (y_0, z_0).

   (c) Use the method of Lagrange multipliers to determine the point on the tangent plane in part (a) that is closest to the origin.

4. **[4+7+4=15 Points]** Do version **A** if n_4 is odd and **B** if n_4 is even, where $\mathsf{s}\,\mathsf{n}_1\mathsf{n}_2\mathsf{n}_3\mathsf{n}_4\mathsf{n}_5\mathsf{n}_6\mathsf{n}_7$ is your student number.

   For $a, b \in \mathbb{R}$ consider the vector field in $\mathbb{R}^3$ defined as

$$\begin{aligned} &\textbf{A:} \qquad \mathbf{F}(x,y,z) = ax\sin(\pi y)\,\mathbf{i} + \left(x^2\cos(\pi y) + bye^{-z}\right)\mathbf{j} + y^2 e^{-z}\,\mathbf{k} \\ &\textbf{B:} \qquad \mathbf{F}(x,y,z) = ax\cos(\pi y)\,\mathbf{i} + \left(x^2\sin(\pi y) + bye^{z}\right)\mathbf{j} + y^2 e^{z}\,\mathbf{k} \end{aligned}$$

   (a) Determine a and b such that $\mathbf{F}$ is conservative.

   (b) For a and b found in part (a), determine a scalar potential for $\mathbf{F}$.

   (c) For a and b found in part (a), compute the line integral of $\mathbf{F}$ along the curve of intersection of the paraboloid $z = x^2 + 4y^2$ and the plane $z = 3x - 2y$ from the point $(0, 0, 0)$ to the point $(1, 1/2, 2)$.

5. [**15 Points**] Do version **A** if n_3 is odd and **B** if n_3 is even, where $\mathsf{s}\,\mathsf{n}_1\mathsf{n}_2\mathsf{n}_3\mathsf{n}_4\mathsf{n}_5\mathsf{n}_6\mathsf{n}_7$ is your student number.

   Verify Stokes' Theorem for

   A: the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $\mathbf{F}(x, y, z) = (yz, -xz, z^3)$ for $(x, y, z) \in \mathbb{R}^3$ and the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the planes $z = 1$ and $z = 3$ with upward-pointing unit normal vector.

   B: the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $\mathbf{F}(x, y, z) = (-yz, xz, z^3)$ for $(x, y, z) \in \mathbb{R}^3$ and the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the planes $z = 1$ and $z = 2$ with downward-pointing unit normal vector.

6. [**15 Points**] Do version **A** if n_2 is odd and **B** if n_2 is even, where $\mathsf{s}\,\mathsf{n}_1\mathsf{n}_2\mathsf{n}_3\mathsf{n}_4\mathsf{n}_5\mathsf{n}_6\mathsf{n}_7$ is your student number.

   Consider the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$
\begin{aligned}
\textbf{A:} \qquad & \mathbf{F}(x, y, z) = (x^2 + y + 2 + z^2)\,\mathbf{i} + (\mathrm{e}^{x^2} + y^2)\,\mathbf{j} + (3 + x)\,\mathbf{k} \\
\textbf{B:} \qquad & \mathbf{F}(x, y, z) = (x^2 + z + 2 + y^2)\,\mathbf{i} + (\mathrm{e}^{z^2} + y^2)\,\mathbf{j} + (3 + y)\,\mathbf{k}
\end{aligned}
$$

   for $(x, y, z) \in \mathbb{R}^3$. Let $a > 0$. Use Gauss' Divergence Theorem to determine the flux of $\mathbf{F}$ in the outward direction through the part of the sphere $x^2 + y^2 + z^2 = 2az + 3a^2$ which lies

   A: above the plane $z = 0$,

   B: below the plane $z = 0$.

   Hint: First 'close' the surface S in a suitable way. A sketch might be helpful.